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M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021
FIRST SEMESTER
MATHEMATICS - CORE
ANALYSIS I
(for those who joined in July 2017 onwards)

Time : Three hours

Maximum: 75 marks

Part – A (10 X 1 = 10 marks)
Answer all question, choose the correct answer:

1. The set of all subsequential limits of a bounded sequence form a ------- subset of X a) connected b) perfect c) compact d) open
2. If A and B are connected subsets of R with usual metric, then a) $A \cup B$ is connected b) $A \cup B$ is not connected c) $A \cup B$ is connected if and only if $A \cap B = \phi$ d) $A \cup B$ is connected if and only if $A \cap B \neq \phi$
3. If (a_n) is a sequence of positive terms and is such that $a_n > a_{n+1} \forall n$, then a) (a_n) is convergent b) (a_n) is not convergent c) (a_n) converges to 0 d) None of the above
4. Let p > 0. $\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^p\right) =$ a) 0 b) 1 c) e d) -1
5. Let $a_n \geq 0 \forall n$ and $\alpha = \limsup_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$. Then $\sum_n a_n$ converges if α a) > 1 b) = 1 c) < 1 d) > 0.
6. $\sum_n a_n$ converges. $\sum_{n>1}\frac{a_n}{\log n}$ a) need not converge b) diverges c) $\sum_n a_n = \sum_{n>1}\frac{a_n}{\log n}$ d) converges
7. Let $f : X \to Y$ be a monotonic decreasing function. Then the number of discontinuities of second kind is a) 0 b) has to be finite c) atmost countably infinite d) can be uncountably infinite
8. The function $f(x) = \begin{cases} \sqrt{2} & x \text{ is rational} \\ x & \text{otherwise} \end{cases}$ is discontinuous at a) of first kind at $\sqrt{2}$ b) of second kind at $\sqrt{2}$ c) of first kind at $\mathbb{R} - \{\sqrt{2}\}$ d) of second kind at $\mathbb{R} - \{\sqrt{2}\}$
9. Let f be a differentiable function. Then the number of simple discontinuities of f' is a) not finite b) 0 c) atmost countably infinite d) can be uncountably infinite
10. Let f be defined for all real numbers and suppose that for all real numbers x, y $|f(x) - f(y)| \leq (x-y)^2$, then a) f is constant b) f is monotonically increasing c) f is monotonically decreasing d) none of the above

Part – B Answer (a) or (b) in each question $(5 \times 5 = 25 \text{marks})$

11a) Define the terms open set, closed set. A set E is open if and only if its complement is closed. Prove. (or)
b) Define closure $\overline{E}$ of a set E. Prove that $\overline{E}$ is the smallest closed set containing E.

12a) If p > 0 and α is real, prove that $\lim_{n\to\infty}\left(\frac{n^\alpha}{(1+p)^n}\right) = 0$.

(or)

b) Define the terms convergent sequence and Cauchy sequence. Prove that convergence sequence is a Cauchy sequence and the converse holds if the space is compact.

13a) State and prove Merten's theorem. (or)

b) State and prove root test.

14a) Define discontinuities of first kind and second kind. Give examples. (or)

b) i) Let X, Y be two metric spaces. Let $f : X \to Y$ be a continuous mapping. Then prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every subset E of X. ii) Also prove that this inclusion can be proper.

15a) State and prove Generalised mean value theorem. Also state Taylor's theorem (or)

b) State and prove L'Hospital's rule.

Part – C Answer (a) or (b) in each question $(5 \times 8 = 40\text{marks})$

16a) Define perfect set. Prove that any nonempty perfect set is uncountable. Deduce any interval in R is uncountable. (or)

b) Let $E \subseteq \mathbb{R}^k$. Prove that the following statements are equivalent. i) E is closed and bounded ii) E is compact iii) Every infinite subset of E has a limit point in E. Deduce Weirstrass' theorem.

17a) i) Define a convergent series. ii) Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \ldots$ converges. iii) Discuss the convergence of $\sum \frac{1}{n^p}$ for values of p. (or)

b) i) Define the number e. ii) Prove that $2 < e < 3$. iii) Prove that $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ iv) Prove that the number e is not rational

18a) i) Prove that for any sequence $\{c_n\}$ of positive numbers, $\limsup_{n \to \infty}\left((c_n)^{\frac{1}{n}}\right) \leq \limsup_{n \to \infty}\left(\frac{c_{n+1}}{c_n}\right)$.

ii) Define the terms power series, radius of convergence of a power series iii) what is the relation between the convergence of a power series and its radius of convergence? Prove. (or)

b) i) Define absolute convergence of a series. ii)Prove that absolute convergence implies convergence but not conversely. iii) If $\sum_n a_n = A; \sum_n b_n = B;$ $\sum_n a_n$ converges absolutely and $c_n = \sum_{k=0}^{n} a_n b_{n-k}$, then prove that $\sum_n c_n = AB$.

19a) i) Define uniformly continuous function. ii) Prove that a continuous function defined on a compact metric space is uniformly continuous. (or)

b) Let E be a noncompact set in R. Then prove that i) There exists a continuous function which is not bounded ii) There exists a continuous and bounded function which has no maximum. iii) If E is further bounded, then there exists a continuous function which is not uniformly continuous.

20a) i) State and prove Generalised mean value theorem. ii) Discuss the behaviour of f according as 1) $f'(x) \geq 0$; 2) $f'(x) = 0$; 3) $f'(x) \leq 0$. iii) Suppose f is real differentiable on [a, b] and suppose $f'(a) < \lambda < f'(b)$, then there exists x such that a < x < b and $f'(x) = \lambda$. (or)

b) i) State and prove chain rule of differentiation. ii) Let f be defined as $f(x) = \begin{cases} x \sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$. Prove that $f'(0)$ does not exist. iii) Let f be defined as $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$. Prove that $f'(0) = 0$.